The familiar properties of the logarithm follow immediately from the definition, in the usual way.

Derivative of $\log_e x$. The curve for $y = \log_e x$ is the same as the curve for $y = e^x$, reflected in the 45° line of the first quadrant. Hence, the slope of the curve $y = \log_e x$ at any point $x = a$ is the reciprocal of the slope of the curve $y = e^x$ at the corresponding point, $y = a$. From this it follows immediately that

$$\left[\frac{d \log_e x}{dx}\right]_{x=a} = \frac{1}{a}.$$

Or, we may write $y = \log_e x$ in the form $e^y = x$, whence $e^y dy = dx$, or $dy = (1/e^y)dx$, or $dy = (1/x)dx$.

Definition of a^x (a positive). Finally, we define a^x (where a is positive) by means of the general equation

$$\log_e (a^x) = x \log_e a, \qquad \text{or} \qquad a^x = e^{x \log_e a}.$$

When x is a positive integer (or a positive or negative rational number) this definition of a^x reduces at once to the forms which are familiar from elementary algebra.

By means of this definition, all the usual properties of a^x follow immediately from the corresponding properties of e^x; in particular, the inverse of the function 10^x gives immediately the logarithm to the base 10 with all its properties. In terms of $\log_{10}$, the definition of a^x may be written in the form

$$\log_{10} (a^x) = x \log_{10} a, \qquad \text{or} \qquad a^x = 10^{x \log_{10} a};$$

which is the form most convenient for numerical computation.

Conclusion. The method of presentation here suggested will be found to be very much shorter and simpler than any of the older methods that give the complete results (see, for example, Chrystal's *Algebra,* or Stolz's *Allgemeine Arithmetik*), and practically as short as many of the methods of the current textbooks, which give the results only for the rational case.

ELEMENTARY PROOF OF A THEOREM DUE TO F. MORLEY.

By TOBIAS DANTZIG, Indiana University.

In a paper read before the Columbus meeting of the American Mathematical Society, December 30, 1915, Professor H. S. White mentioned a theorem due to Professor F. Morley, and first given by him in a memoir entitled: "On Reflexive Geometry."[1] The theorem follows:

If a ring of five circles be formed, the center of each upon a fixed circle and each

[1] *Transactions of the American Mathematical Society,* Vol. 8, 1907, pp. 23–24.

circle of the ring intersecting the next on this fixed circle, the five other intersections when joined in succession will form a pentacle whose vertices lie one upon each of the five circles.[1] See Fig. 2.

Professor Morley's proof of this theorem is based on considerations of synthetic geometry. It is hoped that the very simple elementary proof here given will be of interest; especially as a few other remarkable properties of the same configuration immediately follow from the method used.

1. Let us first recall a theorem of elementary geometry which may not be very well known:

If C_1, C_2 *are centers of two circles intersecting in* A *and* B (Fig. 1) *and if the*

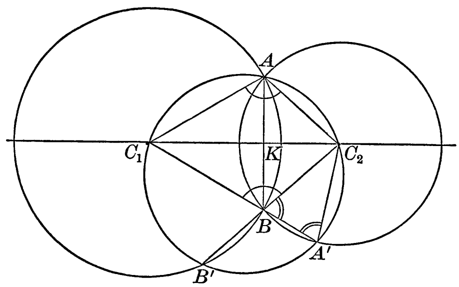

FIG. 1.

two lines C_1B *and* C_2B *be drawn to meet the circles in* A' *and* B', *respectively, the five points* A, A', B', C_1 *and* C_2 *will lie on a circle.*

For $\angle C_1AC_2 = \angle C_1BC_2$ and $\angle C_2BA' = \angle C_2A'B$; hence, we have $\angle C_1AC_2 + \angle C_2A'B = \pi$, and A' is concyclic with A, C_1 and C_2.

Conversely: *If a circle be drawn through the centers of two given circles,* C_1 *and* C_2, *and through one of their intersections say* A, *and if* A', B' *be the two other points where this circle meets the two given circles; then will the three points* C_2, B, B' *lie on a straight line.*

2. With the aid of this theorem Professor Morley's proposition is easily proved. Let C_1, C_2, $\cdots$ (Fig. 2) be the centers of the five circles in question all lying on the circumference O. The circles C intersect in five points A_{12}, A_{23}, $\cdots$ on the circumference O, and in five other points B_{12}, B_{23}, $\cdots$. The pentacle constructed by five joins of the points B has for vertices M_1, M_2, $\cdots$. We are to prove that the points M are on the circumferences C, say M_1 is on C_1, M_2 on C_2 and so on.

Draw C_1P and C_1Q parallel to the sides of the pentacle M_1M_4 and M_1M_3, respectively. Then

$$\angle QC_1B_{12} = \angle M_1B_{12}C_1 = \angle B_{23}B_{12}A_{23} = \angle B_{23}A_{12}A_{23},$$

and

$$\angle PC_1B_{51} = \angle M_1B_{51}C_1 = \angle B_{45}B_{51}A_{45} = \angle B_{45}A_{51}A_{45},$$

[1] This theorem can be considered as the converse of a proposition due to Auguste Miquel and generalized by Clifford. See A. Miquel. Mémoire de Géométrie, *Journal de Liouville,* Vol. X (1844), page 347. Also: Clifford, Collected Mathematical Papers, page 38.

using parallels and inscribed angles subtending equal arcs. Hence C_1P and C_1Q pass through C_4 and C_3, respectively.

Since arc $C_4C_3 = \frac{1}{2}$ arc $A_{45}A_{23}$ it follows that

$$\angle\ C_4C_1C_3 = \angle\ M_1 = \tfrac{1}{2}\ \angle\ B_{51}C_1B_{12}.$$

Now angle $B_{51}C_1B_{12}$ is central in the circle C_1 and angle M_1, subtending the same

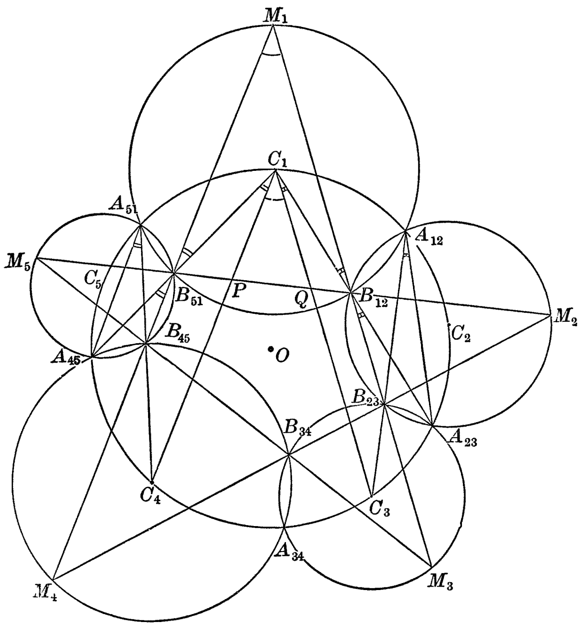

FIG. 2.

arc as the central angle and equal to one half of it, must be inscribed in the same circle. Hence the theorem.

3. By the method used the following results are readily derived:

(1) *The pentacle built on the five centers C has its sides respectively parallel to the pentacle M.*

(2) *The radical axes AB of the five circles C are respectively perpendicular to the sides of the pentacle M.*

(3) *If from any one of the points A we drop perpendiculars on the four adjacent sides of the pentacle M, the feet of the perpendiculars are on one straight line (the Simpson Line). The radical axis through A is perpendicular to the Simpson line. The five angles thus formed at A by the five perpendiculars are respectively equal to the five angles of the pentacle at M, and the Simpson lines are parallel to the sides of the pentacle M.*